stressed conditions. Therefore, in spite of the fact that the method of Neumann (3) strictly mathematically always converges, its numerical application with present computational possibilities in the general case of loading is limited by condition (18). Therefore for $R = (100\text{-}1000)h$ we shall have $d \leqslant (20\text{-}40)h$. This class of shells presents a practical interest.

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ON A MODEL OF A MEDIUM WITH COMPLEX STRUCTURE

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In recent years the attention of a large number of investigators has been drawn to the study of media having complex structure. The simplest of these is the Cosserat medium [1, 2]. Mindlin's medium with microstructure [3] is more complex. An extraordinary complexity is inherent in the multipolar mechanics developed by Green and Rivlin [4].

The essential peculiarity of all these theories is reconsideration of the concept of a point. If in classical continuum mechanics each point possesses only the degrees of freedom of translational displacement, in the Cosserat theory the degrees of freedom of a rigid body are ascribed to it. In the theory of a medium with microstructure each point possesses the degrees of freedom of a body with homogeneous strain, i. e. twelve degrees of freedom. In multipolar mechanics the mechanical state of each point is defined by n kinematical parameters, where n must be finite but may be as large as desired. A new model of a medium of similar type will be constructed below.

We shall postulate the presence of some load-carrying medium and shall assume that its behavior is described by the equations of the classical theory of elasticity

$$(\lambda + \mu)\nabla\nabla \mathbf{u} + \mu\Delta\mathbf{u} - \rho\mathbf{u}^{\cdot\cdot} + \mathbf{K} = 0 \tag{1.1}$$

where ρ is the density of the load-carrying medium, λ and μ are the elastic moduli of the medium, $\mathbf{u}$ is the displacement vector of the points of the load-carrying medium, $\mathbf{K}$ is the external body force (per unit volume). Connected with each point of the load-carrying medium, we shall assume that there is an infinite set of noninteracting, isotropic oscillators with continuously distributed natural frequencies. The equation of motion of a typical oscillator with natural frequency α has the form

$$m(\alpha)\,\mathbf{v}_\alpha^{\cdot\cdot} + c(\alpha)\left[1 + R_\alpha\left(\frac{\partial}{\partial t}\right)\right](\mathbf{v}_\alpha - \mathbf{u}) = \mathbf{Q}_\alpha \tag{1.2}$$

where $\mathbf{v}_\alpha$ is the vector of absolute displacement of the mass of the oscillator and $\mathbf{Q}_\alpha$ is the external force applied to it. The quantity $m(\alpha)\,d\alpha$ is equal to the total mass per unit volume of all the oscillators with natural frequencies lying in the range $(\alpha, \alpha + d\alpha)$. It follows from this that the total mass of all oscillators attached to the particles of a unit volume is

$$m = \int_0^\infty m(\alpha)\,d\alpha \tag{1.3}$$

Further, in Eq. (1.2), the quantity $c(\alpha)$ characterizes the statical stiffness of the suspension of the oscillator. We have by the definition of natural frequency

$$c(\alpha) = \alpha^2 m(\alpha) \tag{1.4}$$

The term with $R_\alpha(\partial / \partial t)$ is introduced into Eq. (1.2) to account for energy dissipation in the suspension of the oscillators. It will be assumed that $R_\alpha(\partial / \partial t)$ is an odd function of the time derivative operator. It will become clear below that consideration of the damping of oscillators is absolutely essential if physically reasonable results are to be obtained. In order to complete the system of equations (1.1) and (1.2), it is necessary to take into account the action of the suspension of the oscillators on the load-carrying medium. The force per unit volume corresponding to this effect is

$$\mathbf{F} = \int_0^\infty c(\alpha)\,[1 + R_\alpha(\partial/\partial t)]\;\mathbf{v}_\alpha - \mathbf{u})\,d\alpha \tag{1.5}$$

This should come into Eq. (1.1) as a body force. Finally, also taking account of the equation of motion of the oscillators, we have

$$(\lambda + \mu)\nabla\nabla\mathbf{u} + \mu\Delta\mathbf{u} - \rho\mathbf{u}^{\cdot\cdot} - \int_0^\infty m(\alpha)\,\mathbf{v}_\alpha^{\cdot\cdot}\,d\alpha + \mathbf{K} + \mathbf{Q} = 0$$

$$m(\alpha)\,\mathbf{v}_\alpha^{\cdot\cdot} + c(\alpha)\left[1 + R_\alpha\left(\frac{\partial}{\partial t}\right)\right](\mathbf{v}_\alpha - \mathbf{u}) = \mathbf{Q}_\alpha \tag{1.6}$$

The vector $\mathbf{Q}$ which occurs here is equal to the external force applied to all the oscillators in a unit volume

$$\mathbf{Q} = \int_0^\infty \mathbf{Q}_\alpha\,d\alpha \tag{1.7}$$

The boundary conditions for the medium which has been introduced are formulated in the same way as in the classical theory of elasticity.

2. Let us consider a slender bar having a free lateral surface. Assuming that a generalized uniaxial stste of stress is present in the bar, we obtain the equations of motion for this special case by the usual methods of the theory of elasticity

$$Eu'' - \rho u^{\cdot\cdot} - \int_0^\infty m(\alpha)\, v_\alpha^{\cdot\cdot}\, d\alpha + K + Q = 0$$

$$m(\alpha)\, v_\alpha^{\cdot\cdot} + c(\alpha)\left[1 + R_\alpha \frac{\partial}{\partial t}\right](v_\alpha - u) = Q_\alpha \tag{2.1}$$

where E is the Young's modulus of the load-carrying medium, u is the displacement of the medium in the direction of the axis of the bar, etc.; the prime denotes diffrentiation with respect to the coordinate x measured along the axis of the bar.

Equations (2. 1), without consideration of the body forces K and Q_α or of the resisting forces was first obtained by Slepian [5]. In that paper the author examines the problem of propagation of stress waves.

In the present work, the problem of a bar of finite length is investigated under the action of either a harmonic force or an impulsive force applied to one end.

3. Let one end of a bar of length l be free and the other be loaded by a harmonic load of frequency ω and unit amplitude. In this case, the boundary conditions have the form

$$x = 0, \quad u' = 0; \qquad x = l, \quad Eu' = e^{i\omega t} \tag{3.1}$$

It is easy to find the steady-state solution of Eqs. (2. 1) for $K = Q_\alpha = 0$ subject to boundary conditions (3. 1). In particular, the values of the accelerations of the points of the load-carrying medium and of the oscillarots turn out to be as follows:

$$u^{\cdot\cdot} = \Phi(\omega, x)\, e^{i\omega t}, \qquad v_\alpha^{\cdot\cdot} = \Psi_\alpha(w, x)\, e^{i\omega t} \tag{3.2}$$

where the transfer functions Φ and Ψ_α may be expressed as

$$\Phi(\omega, x) = \frac{\omega^2 \cos\lambda x}{E\lambda \sin\lambda l}, \quad \Psi_\alpha(\omega, x) = \Phi(\omega, x)\left[1 - \frac{\omega^2}{\alpha^2(1 + i\varphi)}\right]^{-1} \tag{3.3}$$

and the following quantities have been introduced:

$$\lambda^2 = \frac{\omega^2}{E}\left[\rho + \int_0^\infty \frac{c(\alpha)\, d\alpha}{\alpha^2 - \omega^2/(1 + i\varphi_\alpha)}\right], \quad \varphi_\alpha = -iR_\alpha(i\omega) \tag{3.4}$$

φ_α being real.

The variation of the amplitude of vibration along the bar is of chief interest. The squares of the amplitudes of vibration of the load-carrying medium (a^2) and of the oscillators ($b_\alpha)^2$ have the expressions

$$a^2 = |\Phi(\omega, x)|^2, \qquad b_\alpha^{\,2} = a^2\left|1 - \frac{\omega^2}{\alpha^2(1 + i\varphi_\alpha)}\right|^{-2} \tag{3.5}$$

Separating the real and imaginary parts of λ

$$\lambda = \mu - i\eta \tag{3.6}$$

we represent a^2 by the following formula:

$$a^2 = \frac{\omega^4}{E^2|\lambda|^2}\, \frac{\operatorname{ch} 2\eta x + \cos 2\mu x}{\operatorname{ch} 2\eta l - \cos 2\mu l} \tag{3.7}$$

Examination of the numerator of this expression leads to the conclusion that the parameter η determines the rate of decay of the vibrations with distance from their source, and the parameter μ determines how many waves of vibration there are along the bar. Investigation of the denominator shows that the height of the resonant peaks are greater the smaller η is. The width of the peaks on the amplitude-frequency diagram are determined by the parameter μ.

We shall show that the degree of decay of vibrations η does not depend strongly on the damping of the oscillators and remains constant even for vanishingly small damping. For this purpose it is sufficient to show that the expression for λ^2 in Eq. (3.4) remains complex even for zero value damping $\varphi_\alpha \to 0$.

To simplify matters, we shall assume that the damping properties of the suspensions of all the oscillators are the same, i.e. $\varphi_\alpha = \varphi$ does not depend on the parameter α, but can depend on the frequency of the disturbance. But then in order that the free vibrations of each oescillator decay when the load-carrying medium does not move, it is necessary that $\varphi > 0$ for $\omega > 0$. By virtue of the assumption that R and φ are odd functions it follows that for negative ω, φ will also be negative.

We introduce the complex quantity z by the formula

$$z = \omega\,(1 + i\varphi)^{-1/2} \tag{3.8}$$

We take the branch of the radical in (3.8) for which the radical is $+1$ for $\varphi = 0$. It turns out that for $\varphi \neq 0$, the imaginary part of z is negative for all values of frequency ω except zero. Therefore, as $\varphi \to 0$ the complex variable z approaches a real value from the lower half of the complex plane z.

Using the quantity (3.8) which has been introduced, we represent (3.4) in the following form:

$$\lambda^2 = \frac{\omega^2}{E}\left[\rho + \frac{1}{2z}\int\limits_{-\infty}^{\infty}\frac{c(|\alpha|)\,d\alpha}{\alpha - z}\right] \tag{3.9}$$

In the Cauchy integral which occurs here, z is always located in the lower half plane. As $\varphi \to 0$ the complex quantity z approaches points of the real axis along which the integral in (3.9) is taken. In accordance with the Sokhotskii-Plemelj formulas [6] we obtain the following limit of the expression (3.9) as $\varphi \to 0$:

$$\lambda^2 = \frac{\omega^2}{E}\left[\rho + \frac{1}{2\omega}\left(-\pi i c(|\omega|) + \text{v. p.}\int\limits_{-\infty}^{\infty}\frac{c(|\alpha|)\,d\alpha}{\alpha - \omega}\right)\right] \tag{3.10}$$

The integral in this equation is to be interpreted as a principal value. The imaginary part of (3.10) is nonzero. Therefore, the measure of spatial damping of the oscillations η has a finite value even if the damping of the oscillators is taken as small as desired. The quantity η is determined by the variation of stiffness of the suspensions of the oscillators with their natural frequencies. This effect is inherent only in models which take account of the complex structure of the medium, i.e. the presence of suspended oscillators. It is quite unnatural for models of the medium in which a finite stiffness of the suspensions is not considered. From a physical point of view, it can be explained by the fact that the oscillators play the role of dynamic vibration absorbers.

We note that consideration of the damping of the oscillators is necessary for determination of their amplitudes of vibration. The second of Eqs. (3.5) indicates this because in the general case it becomes meaningless for $\varphi = 0$.

4. We shall illustrate the conclusions of the last section by a concrete example. Let

$$m(\alpha) = A\,(\beta^2 + \alpha^2)^{-1} \tag{4.1}$$

where A and β are positive parameters.

Substitution of the expression (4.1) into (1.3) and calculation of the integral results in

$$m = A\pi\,(2\beta)^{-1} \tag{4.2}$$

This equation permits us to eliminate the parameter A from the succeeding equations by expressing it in terms of m and β. We now substitute (4.1) into (1.4) and then into (3.4). As a result, we obtain

$$\lambda^2 = \frac{\omega^2}{E}\left\{\rho + \int_0^\infty \frac{A\alpha^2\,d\alpha}{(\beta^2+\alpha^2)\,[\alpha^2 - \omega^2/(1+i\varphi)]}\right\} \tag{4.3}$$

Calculating the integral which appears in (4.3) by contour integration, we arrive at the following result :

$$\lambda^2 = \frac{\omega^2}{E}\left[\rho + m\left(1 + \frac{i\omega}{\beta\sqrt{1+i\varphi}}\right)^{-1}\right] \tag{4.4}$$

The form of this equation constitutes complete confirmation of the reasoning in the preceding section. Moreover, it indicates the slight dependence of λ on φ for reasonable, and therefore not very large, values of φ.

The example presented is also remarkable for the following reason. If in Eq. (4.4) we set $\rho = 0$ and $\varphi = 0$, we obtain

$$\lambda^2 = \frac{\omega^2}{E}\,\frac{m}{1+i\omega/\beta} \tag{4.5}$$

This is exactly the same structure as in the expression for the square of the wave number for the problem of longitudinal vibrations of a bar made of a Kelvin-Voigt material [7], i.e. for the problem of a bar with finite damping. Thus, the medium which is being studied here has the following interesting peculiarity. Despite the fact that it is "constructed" of elements with low dissipation (high Q) – an ideally elastic load-carrying medium and slightly damped oscillators – it behaves outwardly just like a medium with considerable damping but with simple structure.

At first glance this conclusion seems paradoxical. However, careful investigation shows that considerable energy is actually dissipated even for small damping of the oscillators. This occurs because of the large amplitudes of the oscillators which are at resonance.

5. Let us now study a different loading regime. Let a unit impulse be applied at the section $x = l$. The boundary conditions in this case have the form

$$x = 0, \quad u' = 0, \qquad x = l, \; Eu' = \delta\,(t) \tag{5.1}$$

where $\delta(t)$ is the Dirac delta function.

To simplify the solution of the problem we assume that body forces are absent, $K = Q_\alpha = 0$. Moreover, we shall not take into accout any damping of the oscillators. Considering these simplifications, and performing a Laplace transformation for zero initial conditions, we obtain from Eqs. (2.1) and (5.1)

$$Eu'' - \rho p^2 u - \int_0^\infty m\,(\alpha)\,p^2 v_\alpha\,d\alpha = 0 \tag{5.2}$$

$$m\,(\alpha) p^2 v_\alpha + c\,(\alpha)\;(v_\alpha - u) = 0$$

$$x = 0, \quad u' = 0, \qquad x = l, \quad Eu' = 1 \tag{5.3}$$

where p is the variable of Laplace transformation and the same notation is used for a function and its transform.

The solution of the boundary value problem (5.2), (5.3) is easy to find. In particular, the transformed accelerations of points of the load-carrying medium and of the oscillators are :

$$U = p^2 u = \frac{p^2 \operatorname{ch} \gamma x}{E\gamma \operatorname{sh} \gamma l}, \quad V_\alpha = p^2 v_\alpha = U \left(1 + \frac{p^2}{\alpha^2}\right)^{-1} \tag{5.4}$$

$$\gamma^2 = \frac{p^2}{E}\left[\rho + \int_0^\infty \frac{c(\alpha)\,d\alpha}{\alpha^2 + p^2}\right] \tag{5.5}$$

The inverse transforms for the accelerations U and V_α are found in the general case by the Mellin inversion formulas

$$U = \frac{1}{2\pi i} \int_L \frac{e^{pt} p^2 \operatorname{ch} \gamma x}{E\gamma \operatorname{sh} \gamma l}\,dp, \; V_\alpha = \frac{1}{2\pi i} \int_L \frac{e^{pt} p^2 \operatorname{ch} \gamma x\, dp}{E\gamma \operatorname{sh} \gamma l\,(1 + p^2/\alpha^2)} \tag{5.6}$$

The process of computing these integrals depends greatly on the actual form of the function $\gamma(p)$. In the following section, this calculation will be carried out for the simplest case.

6. Let the function $m(\alpha)$ be given by Eq. (4.1). Substituting (4.1) into (1.4) and then into (5.5), we obtain the following expression for $\gamma(p)$:

$$\gamma^2 = \frac{p^2}{E}\left[\rho + m\left(1 + \frac{p}{\beta}\right)^{-1}\right] \tag{6.1}$$

Inasmuch as the functions (5.4) are meromorphic functions of γ and are therefore, by virtue of Eq. (6.1), single-valued functions of p, the inversion integrals (5.6) can be calculated in this case by using a series expansion in the form of the sum of residues of the integrands in Eqs. (5.6).

The poles of the integrand for U are determined by the equation

$$\operatorname{sh}\gamma l = 0 \tag{6.2}$$

We find from this

$$\gamma l = ik\pi \tag{6.3}$$

where k is an integer, $k = 0, 1, 2, \ldots$.

Substituting (6.3) into (6.1) we find a series of equations for the determination of p

$$-\left(\frac{k\pi}{l}\right)^2 = \frac{p^2}{E}\left[\rho + m\left(1 + \frac{p}{\beta}\right)^{-1}\right] \tag{6.4}$$

For $k = 0$ one negative root and a double root at zero are obtained. The latter can be eliminated from consideration since the expression (5.4) has no pole for $p = 0$.

With the aid of the Hurwitz criterion it is easy to verify that for all other $k \neq 0$ all the roots of Eqs. (6.4) are located in the left half plane. According to (5.4), the function V_α has the same poles as U and, in addition, the two poles $p = \pm i\alpha$. Adding the residues at the poles indicated, we obtain

$$U = \sum_{k,s} \frac{2e^{pt} p^2 \cos(\pi k x / l)}{El(-1)^k (\partial\gamma^2/\partial p)}\Bigg|_{p=p_{ks}} \tag{6.5}$$

$$V_\alpha = \sum_{k,s} \frac{2e^{pt} p^2 \cos(\pi k x / l)}{El(-1)^k (\partial\gamma^2/\partial p)(1 + p^2/\alpha^2)}\Bigg|_{p=p_{ks}} + \alpha\Phi(\alpha, x)\sin\alpha t \tag{6.6}$$

where $\Phi(\alpha, x)$ is given by Eq. (3.3) and the derivative of γ^2 is

$$\frac{\partial\gamma^2}{\partial p} = \frac{p}{E}\left[2\rho + m\left(1 + \frac{p}{\beta}\right)^{-1} + m\left(1 + \frac{p}{\beta}\right)^{-2}\right] \tag{6.7}$$

In Eqs. (6.5) and (6.6) the summation is carried out with respect to k, and for each k a sum is taken on the roots p_{ks} of the characteristic equation (6.4).

The expressions (6.5) and (6.6) permit us to draw the following conclusions of a general nature. Inasmuch as all the roots p of the characteristic equations (6.4) are located in the left half plane (except for the zero roots, which are of no interest), the sums in the expressions for U and V_α involve decaying vibratory solutions. Some time after the application of the impulse these components decay and only the free vibrations of the oscillators can be observed (these are determined by the last term in (6.5)). Comparison of this term with the expression (3.2) shows that the distribution of amplitudes of vibration of the oscillators along the bar coincides, except for a factor, with the distribution of the amplitudes of vibration of the points of the load-carrying medium for a steady excitation with frequency α.

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VARIATIONAL FORM OF THE EQUATIONS OF THE THEORY OF THERMODIFFUSION PROCESSES IN A DEFORMABLE SOLID

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The variational equation corresponding to the fundamental equations of thermoelasticity has been examined by Biot [1, 2], Balabukh and Shapovalov [3], and others [4, 5]. Sedov [6] and his disciples [7, 8] used variational methods to construct new models of continua.

A variational equation equivalent to the system of governing equations of a model which allows description of the interconnection between the deformation, heat and matter diffusion processes, and the most widespread types of boundary conditions, is presented herein.

1. Formulation of the question. Variational equation of the model. A deformed solid representing a two-component solid solution will be